SOLVING FUZZY GRAPH COLORING PROBLEM USING CODED AND UNCODED TECHNIQUES

Synopsis

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1. Background and motivation

A classical graph represents a classical relation between objects. The objects are represented by vertices and relations by edges. A graph coloring problem is one of the most studied problems in discrete and combinatorial optimization. An evidence of this can be found in various papers and books on graph coloring along with associated problems and conjectures.

The graph theory origin can be traced back to Euler's work on the Königsberg bridges problem in 1735. Graphical coloring dates back to at least 1852, when Francis Guthrie came up with the four color conjecture. A classical graph coloring is an assignment of a color to each of the vertices or edges or both in such a way that no two adjacent vertices and incident edges share the same color. Many applications to real world problems like scheduling [26], wiring printed circuits [11], resource allocation, telecommunications, bioinformatics, management sciences, etc. can be modeled by appropriate crisp graphs and solved by coloring of these graphs.

But in real life situations, the decision to be taken must be characterized on the basis of following facts:

(i) Real world data may involve uncertainty, which gives rise to uncertain data,

(ii) Two adjacent vertices, assigned different colors, must remain feasible for all meaningful realizations of the data,

(iii) Solutions must be robust i.e. they must remain feasible even for relatively small changes in the nominal data.

These types of uncertainties cannot be handled in classical logic. There can be vagueness in the description of the objects or in its relationships or in both. For example, the examination scheduling problem can be modeled as a graph where each exam is a vertex of the graph and the edges link those incompatible exams in the sense that there is at least one student sharing both of them. But in real cases, the degree of incompatibility may be low, medium, high etc. which depends upon various parameters other than sharing a common student. If the relations among exams are to be measured in terms of incompatibility degree involving linguistic terms such as low, medium, high, etc., fuzziness should be used in the representation of the graph.

One of the meanings attributed to the term "uncertainty" is "vagueness". To handle such uncertainties and vagueness, the concept of fuzzy sets was introduced by Zadeh [42] in 1965. A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set. This grade corresponds to the degree to which that individual is similar or compatible with the concept represented by the fuzzy set. Formally, a fuzzy set A over universal set X is represented by a membership function $A: X \to [0,1]$, where the transition from membership to non membership is gradual rather than abrupt.

The first definition of fuzzy graph was given by Kaufmann [16] in 1973, based on Zadeh's fuzzy relations [43]. But it was Rosenfeld [30] and Yeh and Bang [41] who laid the foundations for fuzzy graph theory. Generalization of basic concepts of graph theory like paths, cycles, trees, connectedness, and their properties to fuzzy graph theory have been done by Rosenfeld [30]. Fuzzy graph models can represent a complex, imprecise and uncertain problems where classical graph models may fail. Thus, a fuzzy graph representation is more appropriate to reality than crisp graph representation. Fuzzy graph theory has numerous applications in the modern science and technology especially in the fields of information theory, neural network, expert systems, cluster analysis, medical diagnosis, control theory, etc.

The most popular areas of study in graph theory are two types of coloring namely vertex coloring and edge coloring. They have been shown to be computationally difficult and at the same time practically important. Vertex coloring is a function which assigns colors to the vertices so that adjacent vertices receive different colors. The vertex coloring problem is one of the classical NP-hard problems [39], and it is well known not only for its theoretical aspects but also for its difficulty from the computational point of view. A new concept of graph coloring, called a total coloring was introduced by Behzad [2] in 1965. Total coloring is a mixture of the ordinary vertex coloring and edge coloring in such a way that total coloring of a graph *G* assigns a color to each of its vertices and edges. This new concept of graph coloring provides a number of attractive but difficult problems like the famous total coloring conjecture [40].

As fuzzy sets are the generalizations of classical sets, similarly fuzzy graph coloring is the generalization of classical graph coloring. Fuzzy graph coloring has wide applications to many real world problems like traffic light problem [22], Sudoku, political map coloring [33], time table scheduling [27], assignment of jobs, register allocation, frequency assignment, communication networks etc. It has been observed that finding a simple solving algorithm for the coloring of fuzzy graphs still a challenge today. Moreover the problem is NP hard in terms of computational complexity [4]. In this situation, algorithms with effectively less complexity are very useful for graph coloring problems. So, advanced soft computing techniques are useful to keep pace with the development in this field. This motivation forms the basis of research.

2. Preliminaries: Basic definitions

In this section, some basic concepts and definitions in fuzzy theory are summarized, then the concept of fuzzy graph theory and fuzzy graph coloring are introduced.

2.1. Basic concepts: Fuzzy sets

Classical set: A classical set A over a universal set X is defined by a characteristic function $A: X \to \{0,1\},$ defined as

$$A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$

Fuzzy set: A fuzzy set A over a universal set X is defined by a membership function $A: X \to [0,1]$ which maps each $x \in X$ to a real number lying in the closed interval $[0,1],$ where $A(x)$ denotes the membership grade of $x \in X$ in $A.$

$\alpha-$cut and strong $\alpha-$cut: $\alpha-$cut and strong $\alpha-$cut of a fuzzy set A over X are defined as ${}^{\alpha}A = \{x \in X / A(x) \geq \alpha\}$ and ${}^{\alpha+}A = \{x \in X / A(x) > \alpha\}$ respectively.

Support: Let A be a fuzzy set over X. Support of A is defined as $\text{Supp}(A) = \{x \in X \mid A(x) > 0\}$.

Fuzzy set operations: Three important fuzzy set operations comprises of standard fuzzy complement, standard fuzzy intersection and standard fuzzy union.

(i) **Standard fuzzy complement:** The standard fuzzy complement of a fuzzy set A over X is a fuzzy set cA over X defined by a function $c:[0,1]\to[0,1]$, as $c(A(x)) = cA(x), \forall x \in X$, where $cA(x)$ denotes the membership grade of $x \in X$ in fuzzy set cA.

(ii) **Standard fuzzy intersection:** The standard fuzzy intersection of two fuzzy sets A and B over X is a fuzzy set $A \cap B$ over X defined by a binary operation on the unit interval $[0,1]$ as $i:[0,1]\times[0,1]\to[0,1]$, as $i(A(x), B(x)) = (A \cap B)(x), \forall x \in X$, where $(A \cap B)(x)$ denotes the membership grade of $x \in X$ in fuzzy set $A \cap B$.

(iii) **Standard fuzzy union:** The standard fuzzy union of two fuzzy sets A and B over X is a fuzzy set $A \cup B$ over X defined by a binary operation on the unit interval $[0,1]$ as $u:[0,1]\times[0,1]\to[0,1]$, as $u(A(x), B(x)) = (A \cup B)(x), \forall x \in X$, where $(A \cup B)(x)$ denotes the membership grade of $x \in X$ in fuzzy set $A \cup B$.

Fuzzy number: A fuzzy set A over set of real numbers $\mathbb{R}$ defined by the membership function $A:\mathbb{R}\to[0,1]$ is said to be fuzzy number if it satisfies the following properties:

(i) A is a normal fuzzy set,

(ii) α-cuts of A are closed intervals, for $\alpha \in (0,1]$,

(iii) Support of A is bounded.

Linguistic variable: The concept of a fuzzy number plays a fundamental role in formulating quantitative fuzzy variables. These are variables whose states are fuzzy numbers. When in addition,

the fuzzy numbers represent linguistic concepts, such as very small, small, medium, and so on, as interpreted in a particular context, the resulting constructs are usually called linguistic variables.

Each linguistic variable is fully characterized by a quintuple (v, T, X, g, m), where v is the name of the variable, T is the set of linguistic terms of v that refer to a base variable whose values range over a universal set X, g is a syntactic rule (a grammar) for generating new linguistic terms, and m is a semantic rule that assigns to each linguistic term $t \in T$ its meaning, $m(t)$, which is a fuzzy set on X (i.e. $m : T \to \Im(X)$).

Fuzzy binary relation: A fuzzy binary relation R on sets X and Y is a fuzzy set on $X \times Y$ defined by the membership function $R : X \times Y \to [0,1]$ as $R = \{((x, y), R(x, y)) : x \in X, y \in Y\}$.

Fuzzy binary relation on a set: A fuzzy binary relation R on set X is a fuzzy set on $X \times X$ defined by the membership function $R : X \times X \to [0,1]$ as

$R = \{((x, y), R(x, y)) : x \in X, y \in X\}$.

Zadeh [42] introduced fuzzy sets in 1965. His idea based on fuzzy sets can be applied to different areas such as artificial intelligence, natural language, decision making, expert systems, neural networks, control theory, etc. Numerous types of fuzzy sets have been proposed in literature in which some of them focussed on solving the problem of construction of the membership degrees of the elements to the fuzzy set, and others worked on the representation of the vagueness related with the considered problem. Some of them are described briefly as follows:

Interval valued fuzzy set: An interval valued fuzzy set A on a universal set X is defined by a mapping $A : X \to ([0,1])$, where $([0,1])$ denotes the set of all closed subintervals in interval $[0,1]$. In other words, the membership grade of each element of the universe is a closed interval in $[0,1]$ rather than a real number in $[0,1]$.

Intuitionistic fuzzy set: An intuitionistic fuzzy set A on a universal set X is defined as $A=\{(x,\mu_A(x),\nu_A(x)):x\in X\}$, where functions $\mu_A:X\to[0,1]$ and $\nu_A:X\to[0,1]$ defines the degree of membership and non-membership, respectively of an element $x\in X$ in intuitionistic fuzzy set A such that $0\le\mu_A(x)+\nu_A(x)\le 1,\ \forall x\in X$.

$1-(\mu_A(x)+\nu_A(x))$ denotes the degree of hesitation or degree of indeterminacy of $x\in X$ in intuitionistic fuzzy set A.

2.2. Basic concepts: Fuzzy graphs

Graph: A classical graph $G(V,E)$ is a combination of a set of vertices $V=\{v_1,v_2,\ldots,v_n\}$ together with a set of edges $E=\{e_1,e_2,\ldots,e_m\}$, such that each edge e_k is a pair of vertices (v_i,v_j) and is denoted by v_iv_j.

Fuzzy graph: Let V and E denotes the set of vertices and edges of a graph $G(V,E)$ respectively. A fuzzy graph is denoted as $G(A,R)$, where A is a fuzzy set on V and R is a fuzzy relation on $V\times V$ such that $R(u,v)\le\min(A(u),A(v)),\forall u,v\in V$.

Note that, fuzzy set A is defined by the membership function $A:V\to[0,1]$, where $A(v)$ denotes the membership grade of a vertex $v\in V$ in fuzzy set A and fuzzy relation R is defined by membership function $R:V\times V\to[0,1]$, where $R(u,v)$ denotes the membership grade of an edge $(u,v)\in E$ in fuzzy relation R.

Fuzzy subgraph: Let $G(A,R)$ be a fuzzy set with set of vertices V. A fuzzy graph $H(\tilde{V},B,\tilde{R})$ is called a fuzzy subgraph of $G(A,R)$ induced by $\tilde{V}$, if $\tilde{V}\subseteq V$, $B(u)=A(u),\forall u\in\tilde{V}$ and $\tilde{R}(u,v)=R(u,v),\forall u,v\in\tilde{V}$.

Partial fuzzy subgraph: A fuzzy graph $H(B,\tilde{R})$ is called a partial fuzzy subgraph of $G(A,R)$ if $B \subseteq A$ and $\tilde{R} \subseteq R$, i.e. $B(u) \le A(u), \forall u \in V$ and $\tilde{R}(u,v) \le R(u,v), \forall u,v \in V$.

Complement of a fuzzy graph: Complement of a fuzzy graph $G(A,R)$ is a fuzzy graph $\bar{G}(\bar{A},\bar{B})$ where $\bar{A}(u) = A(u), \ \forall u \in V$ and $\bar{R}(u,v) = \min(R(u),R(v)) - R(u,v), \forall u,v \in V$.

Path: A path in a fuzzy graph $G(A,R)$ is a sequence of distinct vertices $u_0, u_1, u_2, ..., u_n$ such that $A(u_{i-1},u_i) > 0, \ i = 1,2,\ldots,n,$ where n denotes the length of the path. Two nodes that are joined by a path are said to be connected. A path with length $n \ge 3$ is said to be a cycle if $u_0 = u_n$.

The strength of a path is defined to be $\underset{i=1}{\overset{n}{\wedge}} A(u_{i-1},u_i)$. In other words, strength of a path is the membership grade of the weakest edge of the path. For any two vertices $u,v \in V$ of a graph $G(A,R)$, the strength of connectedness between u and v is the maximum of the strength of all paths from u to v. It is denoted as $R^{\infty}(u,v)$ and the corresponding path is called the strongest path joining u and v. A fuzzy graph is said to be connected if any two vertices are joined by a path i.e. $R^{\infty}(u,v) > 0, \forall u,v \in V$. Otherwise, it is said to be disconnected.

Complete fuzzy graph: A fuzzy graph $G(A,R)$ is said to be complete if $R(u,v) = \min(A(u),A(v)), \forall u,v \in V^*$, where $V^* = \{v \in V \mid A(v) > 0\}$.

Generalization of classical graphs to fuzzy graphs is given by Kaufmann [16] in 1973. After this, many researchers have worked on fuzzy graphs, types of fuzzy graphs and solved problems of uncertainty and vagueness using fuzzy graphs. Thus, several types of fuzzy graphs can be found in literature. Some of them are as follows:

Interval valued fuzzy graph: A fuzzy graph $G(A,R)$ is said to be an interval valued fuzzy graph if A is an interval valued fuzzy set on V defined by the membership function $A:V\to([0,1]),$ where $A(v)$ denotes the membership grade of a vertex $v\in V$ in interval valued fuzzy set $A,$ i.e. $A=\{(v,A(v)):v\in V\}$ where $A(v)=[A(v)^-,A(v)^+]$ and R is an intuitionistic fuzzy relation on $V\times V$ defined by the membership function $R:V\times V\to([0,1]),$ where $R(u,v)$ denotes the membership grade of an edge $(u,v)\in E$ in intuitionistic fuzzy relation $R,$ $R=\{((u,v),R(u,v)):(u,v)\in E\}$ where $R(u,v)=[R(u,v)^-,R(u,v)^+]$ satisfying $R(u,v)^-\le\min(A(u)^-,A(v)^-)$ and $R(u,v)^+\le\min(A(u)^+,A(v)^+)$ for all $(u,v)\in E.$

Intuitionistic fuzzy graph: A fuzzy graph $G\left(A,R\right)$ is said to be an intuitionistic fuzzy graph if A is an intuitionistic fuzzy set on V defined as $A=\{(v,\mu_A(v),\nu_A(v)):v\in V\},$ where functions $\mu_A:V\to[0,1]$ and $\nu_A:V\to[0,1]$ denotes the degree of membership and non-membership respectively of a vertex $v\in V$ in intuitionistic fuzzy set A such that $0\le\mu_A(v)+\nu_A(v)\le1,\ \forall v\in V$ and R is an intuitionistic fuzzy relation on $V\times V$ defined as $R=\{(u,v),(\mu_R(u,v),\nu_A(u,v)):(u,v)\in E\}$ where functions $\mu_R:V\times V\to[0,1]$ and $\nu_A:V\times V\to[0,1]$ denotes the degree of membership and non-membership respectively of an edge $(u,v)\in E$ in intuitionistic fuzzy relation $R,$ such that $\mu_R(u,v)\le\min(\mu_A(u),\mu_A(v)),$ $\nu_R(u,v)\le\max(\nu_A(u),\nu_A(v))$ and $0\le\mu_R(u,v)+\nu_R(u,v)\le1,$ for all $(u,v)\in E.$

2.3. Fuzzy graph coloring

Graph coloring is one of the oldest and best-known problems of graph theory. Classical graph coloring is mainly based on two types of coloring namely vertex coloring and edge coloring. In 1965, Behzad [2] posed independently a new concept of a graph coloring, called a total coloring. This is a combination of the ordinary vertex coloring and ordinary edge coloring. More work in this regard can

be found in Vizing [36]. In this section, some basic concepts and definitions in classical graph coloring are summarized followed by its generalizations in fuzzy graph coloring.

Classical vertex coloring: A vertex coloring of a graph $G(V,E)$ is a mapping $C:V \to N,$ identifying $C(i)$ is the color of node $i \in V,$ in such a way that two adjacent nodes cannot share the same color i.e. $C(i) \neq C(j)$ if $(i, j) \in E.$

A $k-$vertex coloring C_k of $G(V,E)$ is a coloring function $C_k : V \to k,$ with no more than k different colors. A graph is $k-$vertex colorable if it admits a $k-$vertex coloring. The least value of k such that $G(V,E)$ is $k-$vertex colorable is known as the chromatic number $\chi(G)$ of vertex coloring.

Classical edge coloring: An edge coloring of a graph $G(V,E)$ is a mapping $C:E \to N,$ identifying $C(e_i)$ as the color of edge $e_i \in E,$ in such a way that two adjacent edges cannot share the same color, i.e. $C(e_i) \neq C(e_j)$ whenever edges e_i and e_j are adjacent in $G(V,E).$

A $k-$edge coloring C_k of $G(V,E)$ is a coloring function $C_k : E \to k,$ with no more than k different colors. A graph is $k-$edge colorable if it admits a $k-$edge coloring. The least value of k such that $G(V,E)$ is $k-$edge colorable is known as the chromatic number $\chi^E(G)$ of edge coloring.

Classical total coloring: A total coloring of a graph $G(V,E)$ is a mapping $C:V \cup E \to N,$ where N is a non-empty set of colors. C is said to be total coloring if not only the adjacent vertices, and the adjacent edges have different colors, but also the color of any edge is different from the colors of its two end-points.

A graph $G(V,E)$ is $k-$total colorable if it admits a $k-$total coloring. The least value of k such that $G(V,E)$ is $k-$total colorable is known as total chromatic number $\chi^T(G)$ of total coloring.

Fuzzy vertex coloring: A fuzzy vertex coloring of a fuzzy graph $G(A,R)$ is a family $C=\{C_1,C_2,\ldots,C_k\}$ of fuzzy sets on V satisfying the following properties:

(i) $\underset{i=1}{\overset{k}{\cup}} C_i = A,$

(ii) $C_i \cap C_j = \phi,\ \forall i, j = 1,2,\ldots,k, i \neq j,$

(iii) For every strongly adjacent vertices u,v of $G(A,R)$, $\min\{C_i(u), C_i(v)\}=0, \forall i = 1,2,\ldots,k.$

The least value of k for which the fuzzy graph $G(A,R)$ admits a $k-$fuzzy vertex coloring is known as vertex chromatic number of a fuzzy graph. It is denoted by $\chi_F(G).$

Fuzzy edge coloring: A fuzzy edge coloring of a fuzzy graph $G(A,R)$ is a family $C=\{C_1,C_2,\ldots,C_k\}$ of fuzzy sets on $V\times V$ satisfying the following properties:

(i) $\underset{i=1}{\overset{k}{\cup}} C_i = R,$

(ii) $C_i \cap C_j = \phi,\ \forall i, j = 1,2,\ldots,k, i \neq j,$

(iii) For every incident edges

$\min\{C_i(u,v)$: (u,v) are incident edges from the vertex u, $v \in V\}=0,\ i = 1,2,\ldots,k.$

The least value of k for which the fuzzy graph $G(A,R)$ admits a $k-$fuzzy edge coloring is known as edge chromatic number of a fuzzy graph. It is denoted by $\chi_F^E(G).$

Fuzzy total coloring: A fuzzy total coloring of a fuzzy graph $G(A,R)$ is a family $C=\{C_1,C_2,....,C_k\}$ of fuzzy sets on $V\cup(V\times V)$ satisfying the following properties:

(i) $\underset{i=1}{\overset{k}{\max}} C_i(v) = A(v), \forall v \in V$ and $\underset{i=1}{\overset{k}{\max}} C_i(u,v) = R(u,v), \forall (u,v) \in V\times V,$

(ii) $C_i \cap C_j = \phi,\ \forall i, j = 1,\ldots,k, i \neq j,$

(iii) For every adjacent vertices u, v of $G(A, R)$, $\min\{C_i(u), C_i(v)\} = 0, \forall i = 1, 2, \ldots, k$ and for every incident edges

$$\min\{C_i(u, v): (u, v) \text{ are incident edges from the vertex } u, \text{where } u, v \in V\} = 0, \forall i = 1, 2, \ldots, k.$$

The least value of k for which the fuzzy graph $G(A, R)$ admits k – fuzzy total coloring is known as total chromatic number of a fuzzy graph. It is denoted by $\chi_F^T(G)$.

Review of literature

An important point in the gradual development of modern concept of uncertainty was given by Lofti A. Zadeh [42] in 1965, by introducing fuzzy set as an extension of a classical set. A fuzzy set is defined mathematically by assigning to each possible individual in the universe a value, representing its grade of membership, which corresponds to a degree, to which that individual is similar or compatible with the concept represented by the fuzzy set. Basic literature of fuzzy sets, concepts, operations, relations, logic, applications of fuzzy sets, etc. can be found in Klir and Yuan [20] and Zimmermann [44].

The first definition of a fuzzy graph was given by Kaufmann [16] in 1973, based on Zadeh's fuzzy relations [43]. But it was Rosenfeld [30] who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. The concept of fuzzy trees, blocks, bridges and cut nodes in fuzzy graph has been studied in [30]. During the same time Yeh and Bang [41] have also introduced various connectedness concepts in fuzzy graphs and presented applications in clustering analysis and modeling of information networks by obtaining a threshold graph from a fuzzy graph. The fundamental concepts about graph theory can be found in West [38]. It includes basic algorithms, interesting examples and exercises to expand thorough understanding of the structure of graphs and the techniques used to analyze the related problems.

In graph theory, graph coloring is a special case of graph labelling; it is an assignment of labels traditionally called colors to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color; this is called a vertex coloring. Similarly, an edge coloring assigns a color to each edge so that no two

adjacent edges share the same color and a total coloring assigns a color to each vertex and to each edge such that no adjacent vertices, no adjacent edges, and no edge and its end-vertices are assigned the same color. Coloring a fuzzy graph is the most important and interesting generalization of classical graph coloring. More about coloring of classical graphs can be found in [15,34,37]. Fuzzy graph coloring problems are much thought provoking, attractive and recreational, from both theoretical and practical point of view and these problems have been studied by a number of researchers.

Brélaz [5] proposed some efficient new heuristic methods to color the vertices of a graph which depends upon the comparison of the degrees and structure of a graph. Hertz and Werra [13] described tabu search techniques for moving step by step towards the minimum value of a function for graph coloring problems. A tabu list of forbidden movements was kept updated during the iterations to avoid cycling and being trapped in local minima. Using tabu search techniques, optimal colorings of graphs having up to 1000 nodes was obtained and their efficiency was shown to be significantly superior to the famous simulated annealing.

A metaheuristic approach for vertex coloring problem based on an evolutionary algorithm and post optimization phase based on the set covering is discussed in Malaguti et al. [23]. The complete survey of the most important algorithmic and computational results on the vertex coloring problem and its generalizations can be found in Malaguti and Toth [25]. They introduced the classical models for the vertex coloring problem and discussed how these models can be used and possibly strengthened to derive exact and heuristic algorithms for the problem. Computational results on the best performing algorithms proposed in the literature were reported. Malaguti et al. [24] presented an exact algorithm for the solution of vertex coloring problem based on the well-known set covering formulation of the problem.

Bershtein and Bozhenuk [3] studied the coloring of fuzzy graphs and presented definitions and properties of separation degree and fuzzy chromatic set of fuzzy graphs along with the method for finding fuzzy chromatic set. Eslahchi and Onagh [9] defined the fuzzy coloring of a fuzzy graph and generalized the concepts of vertex-strength and chromatic sum of a crisp graph to fuzzy graphs. They also showed that there exists an upper (a lower) bound for the chromatic fuzzy sum of a fuzzy graph.

Dey and Pal [6] considered a simple undirected graph with fuzzy vertices and fuzzy edges. They introduced a coloring function of fuzzy graph (crisp mode) to color all the vertices of graph of G and obtained the chromatic number of graph based on $\alpha-$cuts of graph, which is a fuzzy number. Ananthanarayanan and Lavanya [1] defined the fuzzy chromatic number, chromatic index and fuzzy total chromatic number of a fuzzy graph as fuzzy numbers through the $\alpha-$cuts of the fuzzy graph which are crisp graphs.

Kishore and Sunitha [18] incorporated the concept of chromatic number of fuzzy graphs introduced by Muñoz et al. [27] and later by Eslahchi and Onagh [9] to modify the chromatic number of a graph in terms of chromatic number of its threshold graph, which is a crisp graph. They proposed algorithms to find chromatic number of crisp graph and fuzzy graph and gave an application of the proposed method to find the solution for the banquet problem. Rosyida et al. [31] considered a graph with crisp vertex set and fuzzy edge set and proposed a new approach to determine fuzzy chromatic set of fuzzy graph through its δ-chromatic number.

Keshavarz [17] introduced a new vertex-coloring problem of a fuzzy graph with crisp vertices and fuzzy edges in which membership degrees of a fuzzy edge is interpreted as incompatibility degree of its associated incident vertices. They also introduced the concept of total incompatibility which is defined as the sum of incompatibility degrees of all incompatible edges. Using this concept, unlike other traditional graph coloring problems, two adjacent vertices can be colored using same color. They named these types of vertices and their associated edge as incompatible vertices and incompatible edge, respectively. For finding an optimal $k-$coloring with minimum degree of incompatibility, he formulated a binary programming problem and a hybrid local search genetic algorithm. He also gave an application of the presented approach in cell site assignment problem by modelling it as a fuzzy graph coloring problem.

Rosyida et al. [32] considered an uncertain graph in which the edges are indeterminate and the existence of edges are characterized by belief degrees which are uncertain measures. They gave the concept and discussed some properties of $\alpha-$cut of uncertain graph and $\alpha-$cut chromatic number. They also gave a real-life decision making problem to illustrate the application of the uncertain

chromatic set and the effectiveness of the uncertain chromatic algorithm. Tahmasbpour and Borzooei [35] introduced different approaches to find edge and vertex chromatic number of a bipolar fuzzy graph, first approach based on the $\alpha-$cuts of a bipolar fuzzy graph and the second approach based on the definition of Eslahchi and Onagh [9] for chromatic number of a fuzzy graph.

Total coloring of fuzzy graphs can be found in Lavanya and Sattanathan [21]. They considered fuzzy graphs with fuzzy set of vertices and fuzzy set of edges and defined fuzzy total coloring in terms of a family of fuzzy sets satisfying certain conditions and hence obtained fuzzy total chromatic number of the graph. Poornima and Ramaswamy [29] studied total coloring of fuzzy graph and analogous to vertex and edge coloring of a fuzzy graph, they introduced the concepts of (d, f) extended $k-$coloring and developed an algorithm for determining (d, f) total chromatic number of a fuzzy graph.

Coloring of fuzzy graphs plays a vital role in theory and practical applications. Gómez et al. [12] discussed an application of fuzzy graph coloring in image classification. They presented a crisp *g*rey coloring algorithm based upon a sequential application of a basic black and white binary coloring procedure. The image was conceived as a fuzzy graph defined on the set of pixels where fuzzy edges represent the distance between pixels. They obtained a more flexible hierarchical structure of colors, which gives useful hints about classes with unclear boundaries.

Dey and Pal [7] gave an application of coloring a simple connected undirected fuzzy graph in traffic control. They represented a traffic network of a city as a fuzzy graph model and discussed a method to find the different type of accidental zones in a traffic flows. Depending on the possibility of accident, they classified the accidental zone of a traffic flow into three type's namely α-strong, β-strong and δ-strong accidental zones which helps to minimize the accident. Two different approaches of fuzzy graph coloring for solving major problems in traffic control were discussed. First approach was based on fuzzy coloring of fuzzy graphs and fuzziness of vertices to minimize the total waiting time of a traffic flow and reducing the traffic jam of a road. Second approach was based on the vertex coloring function of a fuzzy graph (crisp mode) based on $\alpha-$cuts of fuzzy graph G.

Similar applications of fuzzy graph coloring in traffic problem using edge coloring can be found in [10,14,19,28]. A new concept of coloring of fuzzy graphs can be found in Samanta et al. [33]. They gave an application of their proposed technique to color world political map mentioning the strength of relationship among the countries and in traffic light system. Dey et al. [8] proposed a fuzzy graph model to represent the examination scheduling problem of a university and introduced a genetic algorithm based method to find the robust solution of the scheduling problem that remains feasible and optimal or close to optimal for all scenarios of the input data. Fuzzy graph coloring method was used to compute the minimum number of days to schedule the examination.

Munoz et al. [27] gave an application of coloring fuzzy graphs based on the successive coloring of $\alpha-$cuts of the fuzzy graph in the traffic lights problem. They also gave an approach based on an extension of the concept of coloring function by means of a distance defined between colors along with its application in timetabling problem.

3. Proposed work

The proposed work deals with the problem of coloring of fuzzy graphs. The capability of fuzzy sets to express gradual transitions from membership to non-membership and vice versa has a broad utility. It provides us not only a meaningful and powerful representation of measurement uncertainties, but also a meaningful representation of vague concepts expressed in natural language. Thus, an extension of crisp to fuzzy graphs has a broader area of application to real life problems. A number of researches have been carried out on the subject, and numerous theories on the coloring of fuzzy graph have been proposed. Various methods have been used in the coloring of crisp graphs and in the coloring of fuzzy graphs.

The fuzzy graph coloring problem is one of the combinatorial optimization problems. There are several different types of fuzzy graphs. In literature it has been observed that the methods of coloring fuzzy graphs changes among its variants. Thus, coloring problem of different types of fuzzy graphs is one of the most broadly studied problems in the area of fuzzy graph theory. As the problem of coloring requires minimum number of colors to be used or to minimize the total incompatibility

degree of a fuzzy graph, so we can associate optimization problem to it. Thus fuzzy graph coloring problem is a combinatorial optimization problem. Moreover, many real life problems can be modelled as fuzzy graph coloring problems.

No metaheuristic algorithm has been developed to solve fuzzy graph coloring problems except one [17]. In this work, designing and development of metaheuristics for the fuzzy graph coloring problems are intended to be explored. This may involve developing graph specific construction heuristics in order to determine the coloring of a fuzzy graph. Local improvement strategies will be another focus area as they play a key role in the performance of a metaheuristics. The developed construction heuristics/operators/ techniques will be statistically evaluated on benchmark problems/random fuzzy graphs and will also be tested on standard fuzzy graphs with known results (if available). Some applications of fuzzy graph coloring problems in real life applications may also be explored.

Mentioned below are some research directions we intend to consider:

(i) There are several ways in which a graph can be fuzzy. We intend to consider following types of fuzziness embedded in crisp graphs or any of its combination as discussed in Blue et al. [4].

Type I: Fuzzy set of crisp graphs: This case occurs when a fuzzy set G is considered over the set X of crisp graphs G_i, $i = 1, 2, \ldots, n,$ with μ_i being the membership grade of the crisp graph G_i in the fuzzy set G.

Type II: Crisp vertex set and fuzzy edge set: This type occurs when given graph $G(A, R)$ has known vertices but unknown edges. Hence in this type, vertices of $G(A, R)$ are crisp and edges are fuzzy as $V = \{v_1, v_2, ..., v_m\}$ and $E = \{e_1, e_2, ..., e_n\}$ with membership grade on each edge is $\mu = \{\mu_1, \mu_2, ..., \mu_n\}$ respectively and each edge e_i is crisp as its head, tail and weight is fixed.

Type III: Crisp vertices and edges with fuzzy connectivity: When given graph $G(A, R)$ has known vertices and known edges but unknown edge connectivity then this case occurs. Hence in this type, both vertices and edges of $G(A, R)$ are crisp but edges have fuzzy heads and fuzzy tails.

Type IV: Fuzzy vertex set and crisp edge set: This case occurs when given graph $G(A,R)$ has unknown vertices but known edges. Hence in this type, vertices of $G(A,R)$ are fuzzy and edges are crisp as $V=\{v_1,v_2,...,v_m\}$ with membership grade on each vertex is $\mu=\{\mu_1,\mu_2,...,\mu_m\}$ and $E=\{e_1,e_2,...,e_n\}$.

In this case the edge set is crisp even though it depends on fuzzy vertices as edges cannot exist if their head and tail vertices do not exists.

Type V: Crisp graph with fuzzy weights: When given graph $G(A,R)$ has known vertices and known edges but unknown weights on the edges are present then this case occurs. Hence in this type, only weights on graph $G(A,R)$ are fuzzy as $W=\{w_1,w_2,...,w_m\}$.

(ii) Considering different types of fuzzy graphs where the underlying fuzzy set is not an ordinary fuzzy set, but can be its other variants, such as interval-valued fuzzy sets, intuitionistic fuzzy sets, level fuzzy sets, fuzzy rough sets, bipolar fuzzy sets, etc. Thus the study of coloring of fuzzy graphs can be extended to different types of fuzzy graphs.

(iii) Three types of coloring as vertex coloring, edge coloring and total coloring. Moreover, in these types of colorings, the colors itself can also be fuzzy.

(iv) Proposing methods based on uncoded techniques for the fuzzy graph coloring problems.

(v) Methods based on designing and development of metaheuristic for the fuzzy graph coloring problem with larger number of vertices.

(vi) Modelling real life problems such as scheduling problems, network and control problems, allocation problems, etc as fuzzy graph coloring problems.

4. Methodology

Methodology to be adopted for carrying out the above proposed work consists of two main components as follows:

(i) Uncoded techniques: These techniques are based on modified classical techniques based on properties of $\alpha-$cuts.

Given a fuzzy graph $G(A,R)$, a natural approach in order to obtain information about edges and vertices with membership values greater than or equal to α, is to analyze the sequence of the so called $\alpha-$cuts. A fuzzy set A defined on X can be characterized by a family of its $\alpha-$cuts ${}^{\alpha}A=\{x\in X : A(x)\geq\alpha\}, \alpha\in I$, where I denotes the level set. This family of sets is monotonically decreasing, i.e., $\alpha\leq\beta$ implies ${}^{\alpha}A\supseteq{}^{\beta}A, \forall\alpha\in[0,1]$. On the other hand, given a finite monotone family $\{{}^{\alpha}A^{p} : p\in\{1,\ldots,m\}\}$, a fuzzy set A can be defined from the membership function $A(x)=\sup\{\alpha^{p} : x\in{}^{\alpha}A^{p}\}, \forall x\in X$.

Let $\{{}^{\alpha}G(V,{}^{\alpha}E) : \alpha\in I\}$ be the family of α-cuts sets of $G(A,R)$, where the α-cut of a fuzzy graph is the crisp graph ${}^{\alpha}G(V,{}^{\alpha}E)$ with ${}^{\alpha}E=\{(i,j)\in E / R(i,j)\geq\alpha\}$. In other words, $\alpha-$cut of a fuzzy graph $G(A,R)$ is a classical graph ${}^{\alpha}G(V,{}^{\alpha}E)$ consisting of only those edges of $G(A,R)$ with membership grade greater than or equal to α. Thus for $\alpha\in I$, a family of classical graphs ${}^{\alpha}G(V,{}^{\alpha}E)$ can be obtained from fuzzy graph $G(A,R)$. Hence, any classical $k-$coloring function of $G(A,R)$ is defined through this sequence. For each $\alpha\in I$ if ${}^{\alpha}\chi$ denotes the chromatic number of classical graph ${}^{\alpha}G(V,{}^{\alpha}E)$, then the chromatic number of fuzzy graph $G(A,R)$ is the fuzzy number $\chi(G)=\{x,\nu(x)\,|\,x\in X\}$, where $X=1,\ldots,|V|$, $\nu(x)=\sup\{\alpha\in I : x\in{}^{\alpha}A\}, \forall x\in X$ and ${}^{\alpha}A=\{1,2,\ldots,{}^{\alpha}\chi\}, \forall\alpha\in I$.

The chromatic number of a fuzzy graph is a normalized fuzzy number whose modal value is associated with the empty edge-set graph. Its meaning depends on the sense of index α, and it can be interpreted in the following way: for lower values of there are many incompatible links between nodes

and, consequently, more colors are needed in order to consider these incompatibilities; on the other hand, for higher values of α there are fewer incompatible links between nodes and less colors are needed. The chromatic number sums up all this information in order to manage the fuzzy problem.

The fuzzy coloring problem consists of determining the chromatic number of a fuzzy graph and an associated coloring function. In this approach, for any level α, the minimum number of colors needed to color the crisp graph ${}^{\alpha}G(V, {}^{\alpha}E)$ are computed. In this way, the fuzzy chromatic number is defined as a fuzzy number through its $\alpha-$cuts.

(ii) Coded techniques (soft computing techniques): These techniques includes genetic algorithms, memetic algorithms, particle swam intelligence, ant colony optimization, etc.

Generally, metaheuristics based on population follow a similar set of operations and hence they can be defined in a common framework shown in the following psuedocode as:

1. Set no. of iterations $g = 0$
2. Initialize S solutions of P_g population randomly
3. Evaluate each element of P_g
4. Mark the best solution of P_g as b_g^*
5. Repeat
6. Choose r solutions (S_g) from P_g using a selection operator
7. Generate s new solutions (R_g) from S_g using a generation operator
8. Choose k solutions (Q_g) from P_g using a replacement operator
9. Update P_g by replacing Q_g using $k-$ solutions from a group of any combination of at most three sets S_g, R_g and Q_g using an update operator
10. Evaluate each element of P_g
11. Identify the best solution of P_g and update b_g^*

12. $g \leftarrow g+1$

13. Until (a termination condition satisfied)

14. Declare the optimal solution as b_g^*

Some of the population based metaheuristics are described briefly as follows:

Genetic Algorithm

The heuristics search and optimization techniques that mimic the process of natural evolution is called genetic algorithm. This type of evolutionary algorithm is based on the principle of evolution via natural selection with crossover and mutation operators on a population size. It acts on a population of individuals that undergo selection in the presence of operators such as mutation and crossover. Then a fitness function is used to evaluate the individuals. In this algorithm, first initialize random population of chromosomes and then find a fitness function for it. For each iteration, a new population of individuals is devolved from one generation to the next generation through selection, crossover and mutation operators. And then after this, the new population of individuals so obtained, replace the previous one and the overall best solution is updated. This procedure repeats until the desired number of iterations or the best solution finds or the algorithm converges to a solution.

Memetic Algorithm

A class of stochastic global search heuristics in which evolutionary algorithms based approaches for example Genetic Algorithm are combined with problem-specific solvers like local search heuristic techniques, approximation algorithms or sometimes partia) exact methods are called memetic algorithms. There are several names of Memetic algorithms found in literature such as hybrid Genetic Algorithms, Baldwinian Evolutionary Algorithms, Lamarckian Evolutionary Algorithms, genetic local search algorithms, etc. This algorithm is based on population and proved for variety of problems and in particular for finding an approximate solution of NP-hard optimization problems.

Ant Colony Optimization

Ant colony optimization is a metaheuristic optimization technique which was first applied on travelling salesman problem and later on an improved ant colony optimization coded technique was also developed and applied on numerous number of optimization problems. This method is motivated by the randomized search behaviour of real ants and pheromone communication between them during search of food which enables them to find the shortest path between their colony and food source. By performing random walk, ants search the food source and leave pheromone on the trail to mark the path which would guide other ants to find the food source. Other ants smell the pheromone and follow the same path. So, the trail is of high pheromone by which most of the ants go through. Hence, they follow the path of high pheromone while returning to their nests. So, in this way they find their shortest path from nest to food source. This algorithm starts with a random population and pheromone parameter P and in each iteration, the value of pheromone is updated by number of steps. And at last a solution with high pheromone value is selected.

Particle Swarm Optimization

The coded technique particle swarm optimization is a swarm intelligence metaheuristic motivated by the common manners of animals that don't have any leader in their group (or swarm). For example, bird flocks have no leader but they find their path for getting food by following their members of the group that has the closest position to the source of food. In this method, animals find their best solutions through communication between members who already have a better solution and also animals which have better solution informs to his crowd and others also get the same solution by following him. This process repeats until best solution discovered. Particle swarm optimization is different from genetic algorithms in the sense that it does not change its population in each iteration but update the position of members of the population.

Some other metaheuristic technique like artificial bee colony optimization, tabu search, simulated annealing, scatter search, differential evolution, etc. may also be used to solve the fuzzy graph coloring problems.

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